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# START-UP DEMONSTRATION TEST USING FINITE MARKOV CHAIN IMBEDDING

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ABSTRACT. The finite Markov chain imbedding(FMCI) method has been consistently referred to miscellaneous areas to discovering the accurate or approximate distributions of runs and patterns under independent and identically distributed or Markov dependent trials. And a start-up demonstration test is a method by which an alarm equipment explains the credibility of fire alarm systems. This paper deals with a start-up demonstration test using the finite Markov chain imbedding method.

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## 1. INTRODUCTION

A start-up demonstration test is a mechanism to estimate the meaning to successful start-ups. Specifically, the mechanism is estimated by quality or reliability of a unit such as fire alarm systems.

Hahn and Gage [1] proposed a start-up demonstration test acknowledged a unit if many consecutive successful start-ups are observed(CS model). Presuming that the results of the start-ups are independent and identically distributed(i.i.d.) binary random variables, a repetition equation is originated from the waiting period probability until the needed number of consecutive successes are observed.

In addition, Viveros and Balakrishnan [2] also proposed the CS model by presuming that the results of the independent start-ups are Markov-dependent binary random variables in place of independent and identically distributed binary trials; the instants of the waiting time until close and rational methods for the probability of success have been progressed. Balakrishnan, N., Balasubramanian, K., Viveros, R. [3] proposed the mutual probability generating function(p.g.f) of start-up demonstration tests considering Markov reliant binary results. At once, the rule of accurate deeds has been proposed to the instance of independent results. Balakrishnan, Mohanty and Aki [4] proved the CS demonstration test under the set of Markov dependence. It is valued to refer that the corresponding waiting time problems have been proposed to the term *geometric distribution of order k*.

Fu and Koutras [5] developed the finite Markov chain imbedding(FMCI) method for study the accurate distributions for the number of particular runs and patterns in a series of Bernoulli trials. It has been proven in the research paper that the FMCI method has been consistently used for various problems, for example, reliability (Cui, L., Xu, Y., Zhao, X. [6]), borderline intersecting trouble (Fu and Wu [7]), quality control (Chang and Wu [8]). Specially, the FMCI method has been proven for use of several typical random permutation troubles, such as number of continuations (Fu [9]), Eulerian and Simon Newcomb numbers (Fu, J. C., Lou, W. Y. W., Wang, Y. J. [10]); but using combinatorial technique can be very monotonous at times. Therefore, FMCI method is frequently performed as an alternative to the combinatorial technique for discovering the accurate distributions.

The essential idea of FMCI depends upon the Chapman-Kolmogorov equation. It is whose main theory counts upon being feasible for imbed or switch a statistic or a random variable for a finite Markov chain. Moreover, the distribution of the statistic is acquired through transition matrices for the imbedded Markov chain. In numerous troubles, the imbedding systems of switching non-Markov random variables to Markov chains are carried through on an independent basis. Fu [11] expanded a system named *"forward and backward principle(FBP)"* to intentionally implement the imbedding system and accomplished the accurate distributions to widespread runs and patterns from multi-state trials. The FBP proves how to make a Markov chain to imbeded for taking all the data contained at the initial random variable or statistic.

The rest of this paper is organized in the following ways. Section 2 introduces the finite Markov chain imbedding method. Section 3 introduces the runs and patterns of a sequence of two-state trials for waiting-period distribution of a successful run. Section 4 introduces the general solution for CS model by the finite Markov chain imbedding method of discovering the accurate distribution of runs.

## 2. Finite Markov chain imbedding

The finite Markov chain imbedding(FMCI) method of discovering the distribution of the random variable $X_n(\Lambda)$ has its initial beginnings of a succession of research papers by Fu [12], [13], Fu and Hu [14], Chao and Fu [15], [16] and Fu and Lou [17]. The "finite Markov chain imbeddable" named for explaining a random variable made know by Fu and Koutras [5].

Let it be an index set to $\Gamma_n = \{0, 1, \ldots, n\}$, and let it be a finite state space for $\Omega = \{a_1, a_2, \ldots, a_m\}$.

**Definition.** The non-negative integer used random variable $X_n(\Lambda)$ is *finite Markov chain imbeddable* if :

(a) occurs a finite Markov chain $\{Y_t : t \in \Gamma_n\}$ defined on a finite state space $\Omega$ with initial probability vector $\boldsymbol{\xi}_0$,

(b) occurs a finite division $\{C_x : x = 0, 1, \ldots, l_n\}$ on the state space $\Omega$, and

(c) for all $x = 0, 1, \ldots, l_n$, we have

$$P(X_n(\Lambda) = x) = P(Y_n \in C_x | \boldsymbol{\xi}_0). \tag{1}$$

Let it be the succession of $m \times m$ transition probability matrices of the finite Markov chain $\{Y_t\}$ defined on the state space $\Omega$ with beginning probability distribution $\boldsymbol{\xi}_0 = (P(Y_0 = a_1), P(Y_0 = a_2), \ldots, P(Y_0 = a_m))$ for $\{\mathbf{M}_t\}_{t=1}^n$.

Fu and Koutras [5] proposed the following Theorem 2.1.

**Theorem 2.1.** *If* $X_n(\Lambda)$ *is* finite Markov chain imbeddable,

$$P(X_n(\Lambda) = x) = \boldsymbol{\xi}_0(\prod_{t=1}^{n} \mathbf{M}_t)\mathbf{U}'(C_x), \tag{2}$$

where $U(C_x) = \Sigma_{r:a_r \in C_x} \boldsymbol{e}_r$, $\boldsymbol{e}_r$ is a $1 \times m$ unit row vector matching from state $a_r$, $\boldsymbol{\xi}_0$ is *the beginning probability vector*, and $\mathbf{M}_t$, $t = 1, \ldots, n$, are *the transition probability matrices of the imbedded Markov chain.*

## 3. Runs and patterns in a sequence of two-state trials

3.1. **Geometric distribution of order** $c$ **- I.I.D. trials.** Assume that the sequence of trials, $X_1, X_2, \ldots$, are independent and identically distributed with success (S) probability $p = P(X_i = 1)$ and failure (F) probability $q = 1 - p = P(X_i = 0)$. In this case, the distribution of the discontinuing time variable $T_c$ observed to a sequence of $c$ consecutive successes for the first time is mentioned in as the *geometric distribution of order* $c$.

Feller [18] proposed a practical use of the theory about repeated event and testified that the distribution of the trial number $n$ has probability generating function when the one run of $c$ happens.

$$G(z) = \frac{p^c z^c (1 - pz)}{1 - z + qp^c z^{c+1}}, \tag{3}$$

where $0 < p < 1$, $n = c, c+1, \ldots$, and $c = 1, 2, \ldots$, respectively.

The mean and variance is

$$E(X) = \frac{1 - p^c}{qp^c}, \quad V(X) = \frac{1}{(qp^c)^2} - \frac{2c+1}{qp^c} - \frac{p}{q^2}. \tag{4}$$

The diverse relevant geometric distributions of order $c$ are argued with the following substance. Let $X_1, X_2, \ldots$ be a succession of binary trial deriving in the probability of success $p = Pr[X_i = 1]$ respectively. Let $T_c$ be the waiting time until a $c$ consecutive success happens to the one time :

$$\begin{aligned} T_c &= \min\{n : X_{n-c+1} = \cdots = X_n = 1\} \\ &= \min\{n : \prod_{i=n-c+1}^{n} X_i = 1\} \\ &= \min\{\sum_{i=n-c+1}^{n} X_i = c\}. \end{aligned} \tag{5}$$

Philippou and Muwafi [19] proposed that

$$f(x) = \sum \binom{x_1 + x_2 + \cdots + x_c}{x_1, x_2, \ldots, x_c} p^x \left(\frac{q}{p}\right)^{x_1 + \cdots + x_c}, \tag{6}$$

where $x \geq c$, where the sum is carried out all non-negative integers $x_1, x_2, \ldots, x_c$ dependent to the condition $\sum_{i=1}^{c} i x_i = x - c$.

Uppuluri and Patil [20] proposed the easier equation with implies binomial coefficients.

$$f(x) = p^c \sum_{j=0}^{\infty} (-1)^j \binom{x-c-jc}{j} (qp^c)^j - p^{c+1} \sum_{j=0}^{\infty} (-1)^j \binom{x-c-jc-1}{j} (qp^c)^j, \tag{7}$$

where $x \geq c$.

They derived from this expression by enlarging the probability making function $G(z)$ of the distribution of a Taylor series roughly $z = 0$. Other formula implying once more only the binomial coefficients is

$$f(x) = \sum_{i=1}^{x-c} q^i p^{x-i} \sum_{j=0}^{[\frac{x-i-c}{c}]} (-1)^j \binom{i}{j} \binom{x-c(j+1)-1}{i-1}, \tag{8}$$

where $x \geq c+1$.

Muselli [21] proposed the useful singular summarized equation.

$$f(x) = \sum_{j=1}^{[\frac{x+1}{c+1}]} (-1)^{j-1} p^{jc} q^{j-1} \left\{ \binom{x-jc-1}{j-2} + q \binom{x-jc-1}{j-1} \right\}. \tag{9}$$

3.2. **Waiting-time distribution for a successful run.** The distribution of $W(\Lambda)$ for Bernoulli trials referred to the geometric distribution of order c has its initial beginnings of a succession of research papers by Aki [22] and Hirano [23].

Let it be the uncomplicated system of c consecutive successes for $\Lambda = S \cdots S$, and let it be the random variable $W(\Lambda)$ as the waiting time for system $\Lambda$ to $\Lambda$ to happen, *i.e.*

$$W(\Lambda) = \inf\{n : X_{n-c+1} = X_{n-c+2} = \cdots = X_n = S\} \tag{10}$$

For instance, taken c = 6, $W(\Lambda) = 8$ signifies that the system SSSSSS happens to the initial time after eight trials, as in SFSSSSSS.

Fu and Lou [24] proposed the following Theorem 3.1.

**Theorem 3.1.** *To a taken system length* $c \geq 1$ *and a succession of Bernoulli trials* $\{X_t\}$*, the distribution of* $W(\Lambda)$ *is taken by*

$$P(W(\Lambda) = n) = \boldsymbol{\xi} \mathbf{N}^{n-1}(\Lambda)(\mathbf{I} - \mathbf{N}(\Lambda))\mathbf{1}', \tag{11}$$

where $\boldsymbol{\xi} = (1, 0, \ldots, 0)$ is a $1 \times c$ row vector, and $\mathbf{N}(\Lambda)$ is the $c \times c$ important transition probability submatrix of

$$\mathbf{M}(\Lambda) = \begin{matrix} 0 \\ 1 \\ \vdots \\ \vdots \\ c-1 \\ \alpha \end{matrix} \left( \begin{array}{cccccc|c} q & p & 0 & \cdots & 0 & 0 \\ q & 0 & p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q & 0 & \cdots & \cdots & 0 & p \\ \hline 0 & 0 & \cdots & \cdots & 0 & 1 \end{array} \right)_{(c+1)\times(c+1)} = \left( \begin{array}{c|c} \mathbf{N}(\Lambda) & \mathbf{C} \\ \hline \mathbf{0} & \mathbf{1} \end{array} \right). \tag{12}$$

The probability generating function of $W(\Lambda)$ is taken by

$$\varphi_W(s) = \frac{p^c s^c (1-ps)}{1-s+qp^c s^{c+1}}. \tag{13}$$

Such as several easy calculations, it can indicate from the upper equation that the probability generating function of $W(\Lambda)$ is

$$\varphi_w(s) = 1 + (s-1)\boldsymbol{\xi}(\mathbf{I} - s\mathbf{N})^{-1}\mathbf{1}'. \tag{14}$$

The mean for the waiting time $W(\Lambda)$ is

$$\varphi_W(s) = \boldsymbol{\xi}(\mathbf{I} - \mathbf{N})^{-1}\mathbf{1}'. \tag{15}$$

## 4. CS model

4.1. **The general solution for CS model.** Hahn and Gage [1] proposed a start-up demonstration test in which an instrument under experiment is received if a stipulated number of consecutive successful(CS) start-ups is checked to which referred as CS model. Presume that start-ups are independent events and obeys to next. Then it will get answers to the questions below :

What is the probability that for a certain unit certain CS start-ups ba able to accomplish in certain tried start-ups?
$p$ = the probability of success in a singular tried start-up
$q$ = the probability of failure in a singular tried start-up; $q = 1 - p$
$c$ = the needed number of consecutive success to perform practical use
$P(x)$ = the probability of requiring exactly $x$ tried start-ups for practical use
$R(y)$ = the probability of requiring a sum of $y$ or fewer tried start-ups for practical use $(= \sum_{x=1}^{y} P(x))$

The probability function for the required number of tried start-ups to practical use is as in the following by Hahn and Gage [1].

$$p(x) = \begin{cases} 0 & \text{if } x < c \\ p^c & \text{if } x = c \\ qp^c & \text{if } c+1 \le x \le 2c \\ qp^c[1 - \sum_{t=1}^{t=x-2c} P(c+t-1)] & \text{if } x \ge 2c+1 \end{cases} \tag{16}$$

4.2. **Numerical research for CS model.** In the fire alarm systems, FIGURE 1, 2, 3, 4 compares the effect about $p$ in the needed number of CS start-ups for proof on the entire number of start-ups to accomplish practical use. In FIGURE 1, the probability of accomplishing a successful start-up in a singular trial was p = 0.90, 0.95, 0.99 and the number of CS start-ups needed to practical use was c = 5. The probability of needing a sum of y or fewer tried start-ups to acceptance is proved as the three curves. In FIGURE 2, the probability of accomplishing a successful start-up in a singular trial was p = 0.90, 0.95, 0.99 and the number of CS start-ups needed to practical use was c = 15. The probability of needing a sum of y or fewer tried start-ups for acceptance is proved as the three curves. In FIGURE 3, the probability of accomplishing a successful start-up in a singular trial was p = 0.90, 0.95, 0.99 and the number of CS start-ups needed to practical use was c = 30. The probability of needing a sum of y or fewer tried start-ups for acceptance is proved as the three curves. In FIGURE 4, the probability of accomplishing a successful start-up in a singular trial was p = 0.90, 0.95, 0.99 and the number of CS start-ups needed to practical use was c = 50. The probability of needing a sum of y or fewer tried start-ups for acceptance is proved as the three curves.

FIGURE 5, 6, 7 compares the effect about $c$ in the needed number of CS start-ups for proof on the entire number of start-ups to accomplish practical use. In FIGURE 5, the probability of accomplishing a successful start-up in a singular trial was p = 0.90 and the number of CS start-ups needed to practical use was c = 5, 15, 30, 50. The probability of needing a sum of y or fewer tried start-ups to acceptance is proved as the four curves. In FIGURE 6, the probability of accomplishing a successful start-up in a singular trial was p = 0.95 and the number of CS start-ups needed to practical use was c = 5, 15, 30, 50. The probability of needing a sum of y or fewer tried start-ups to acceptance is proved as the four curves. In FIGURE 7, the probability of accomplishing a successful start-up in a singular trial was p = 0.99 and the number of CS start-ups needed to practical use was c = 5, 15, 30, 50. The probability of needing a sum of y or fewer tried start-ups to acceptance is proved as the four curves.

TABLE 1, 2, 3, 4 shows how the value varies according to p and the tried start-up for proof on the entire number of start-ups to accomplish practical use. In TABLE 1, the probability of accomplishing a successful start-up in a singular trial was p = 0.90, 0.95, 0.99 and the number of CS start-ups needed to practical use was c = 5. This table compares the probability of singular successful start-up for p = 0.90, 0.95 and 0.99 when accomplishing accurately 5 tried start-ups, 15 or fewer tried start-ups, 30 or fewer tried start-ups, 50 or fewer tried start-ups and 75 or more tried start-ups in c = 5. In TABLE 2, the probability of accomplishing a successful start-up in a singular trial was p = 0.90, 0.95, 0.99 and the number of CS start-ups needed to practical use was c = 15. This table compares the probability of singular successful start-up for p = 0.90, 0.95 and 0.99 when accomplishing accurately 15 tried start-ups, 30 or fewer tried start-ups, 50 or fewer tried start-ups, 75 or fewer tried start-ups and 105 or more tried start-ups in c = 15. In TABLE 3, the probability of accomplishing a successful start-up in a singular trial was p = 0.90, 0.95, 0.99 and the number of CS start-ups needed to practical use was c = 30. This table compares the probability of singular successful start-up for p = 0.90, 0.95 and 0.99 when accomplishing accurately 30 tried start-ups, 50 or fewer tried start-ups, 75 or fewer tried start-ups, 105 or fewer tried start-ups and 140 or more tried start-ups in c = 30. In TABLE 4, the probability of accomplishing a successful start-up in a singular trial was p = 0.90, 0.95, 0.99 and the number of CS start-ups needed to practical use was c = 50. This table compares the probability of singular successful start-up for p = 0.90, 0.95 and 0.99 when accomplishing accurately 50 tried start-ups, 75 or fewer tried start-ups, 105 or fewer tried start-ups, 140 or fewer tried start-ups and 180 or more tried start-ups in c = 50.

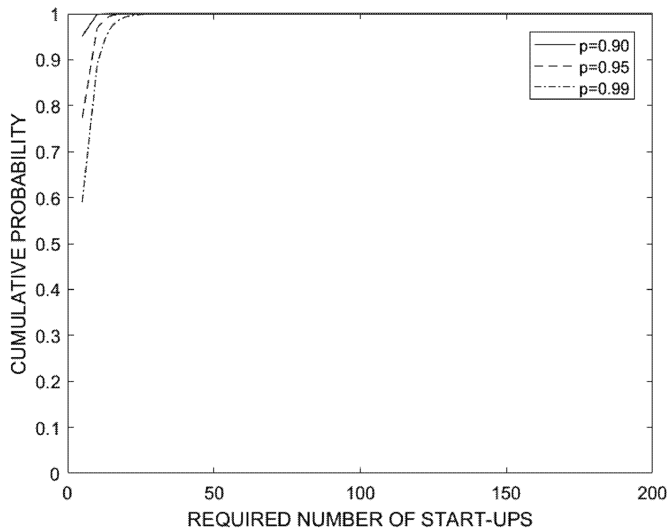


**FIGURE 1.** cumulative probability distribution of entire number of trials to acquire 5 CS start-ups —— Supposing probability of singular successful start-up for p = 0.90, 0.95 and 0.99

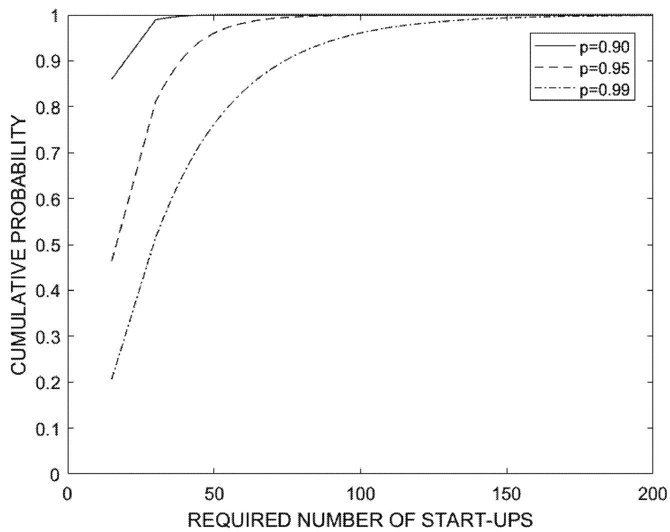


**FIGURE 2.** cumulative probability distribution of entire number of trials to acquire 15 CS start-ups —— Supposing probability of singular successful start-up for p = 0.90, 0.95 and 0.99

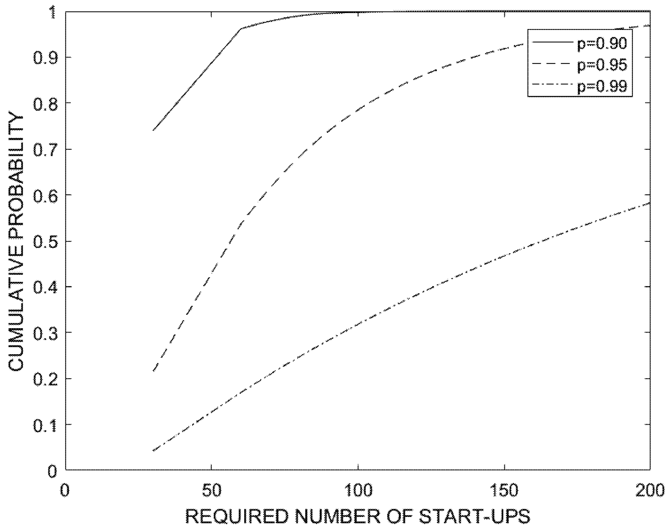


**FIGURE 3.** cumulative probability distribution of entire number of trials to acquire 30 CS start-ups —— Supposing probability of singular successful start-up for p = 0.90, 0.95 and 0.99

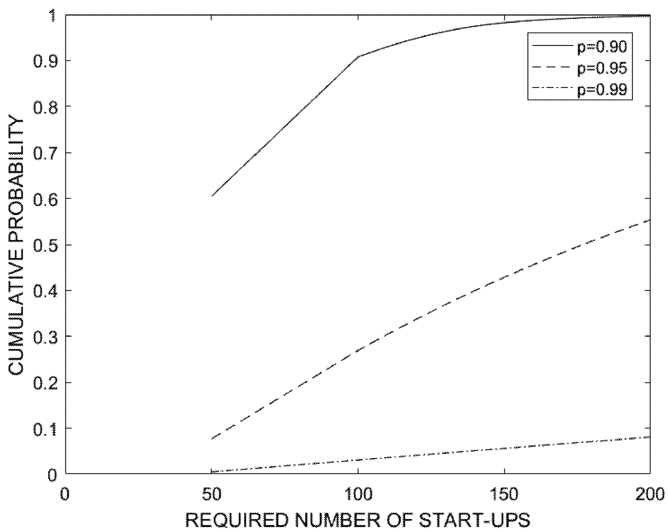


**FIGURE 4.** cumulative probability distribution of entire number of trials to acquire 50 CS start-ups —— Supposing probability of singular successful start-up for p = 0.90, 0.95 and 0.99

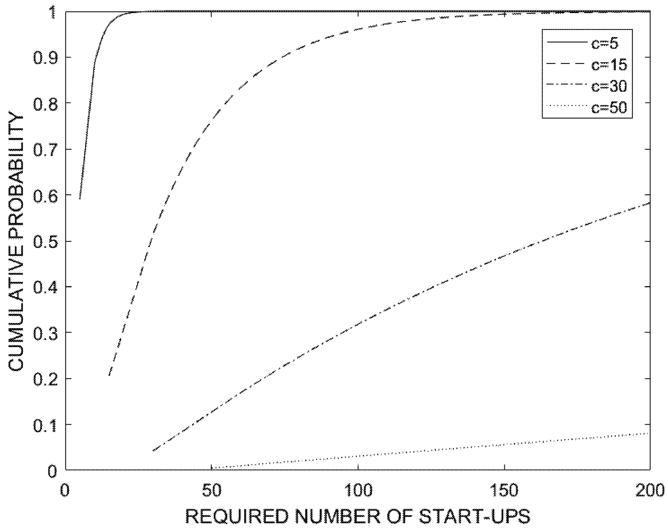


**FIGURE 5.** cumulative probability distribution of entire number of trials to acquire c = 5, 15, 30, 50 CS start-ups —— Supposing probability of singular successful start-up for p = 0.90

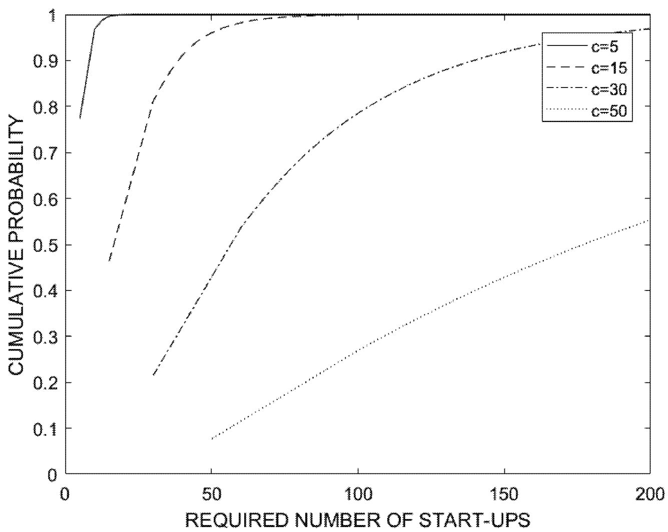


**FIGURE 6.** cumulative probability distribution of entire number of trials to acquire c = 5, 15, 30, 50 CS start-ups —— Supposing probability of singular successful start-up for p = 0.95

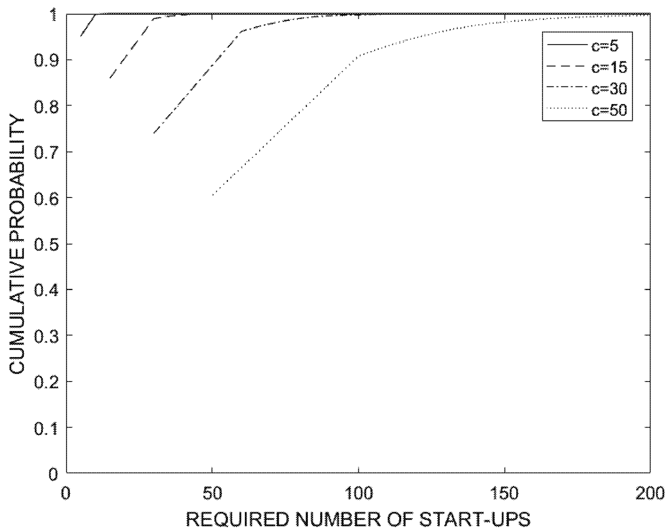


**FIGURE 7.** cumulative probability distribution of entire number of trials to acquire c = 5, 15, 30, 50 CS start-ups —— Supposing probability of singular successful start-up for p = 0.99

**TABLE 1.** probability for a specific unit to accomplish 5 CS start-ups in accurately 5 tried start-ups, 15 or fewer tried start-ups, 30 or fewer tried start-ups, 50 or fewer tried start-ups and 75 or more tried start-ups —— Supposing probability of singular successful start-up for p = 0.90, 0.95 and 0.99

| c = 5 | p = 0.90 | p = 0.95 | p = 0.99 |
|---|---|---|---|
| Accurately 5 Tried Start-Ups [P(5)] | 0.590490 | 0.077378 | 0.950990 |
| 15 or Fewer Tried Start-Ups [P(15)] | 0.971773 | 0.996019 | 0.999966 |
| 30 or Fewer Tried Start-Ups [P(30)] | 0.999630 | 0.999995 | 1.000000 |
| 50 or Fewer Tried Start-Ups [P(50)] | 0.999999 | 1.000000 | 1.000000 |
| 75 or More Tried Start-Ups [1-P(75)] | 1.000000 | 1.000000 | 1.000000 |

**TABLE 2.** probability for a specific unit to accomplish 15 CS start-ups in accurately 15 tried start-ups, 30 or fewer tried start-ups, 50 or fewer tried start-ups, 75 or fewer tried start-ups and 105 or more tried start-ups —— Supposing probability of singular successful start-up for p = 0.90, 0.95 and 0.99

| c = 15 | p = 0.90 | p = 0.95 | p = 0.99 |
|---|---|---|---|
| Accurately 15 Tried Start-Ups [P(15)] | 0.205891 | 0.463291 | 0.860058 |
| 30 or Fewer Tried Start-Ups [P(30)] | 0.514728 | 0.810760 | 0.989067 |
| 50 or Fewer Tried Start-Ups [P(50)] | 0.762145 | 0.960069 | 0.999727 |
| 75 or Fewer Tried Start-Ups [P(75)] | 0.903144 | 0.994591 | 0.999998 |
| 105 or More Tried Start-Ups [1-P(105)] | 0.967043 | 0.999508 | 1.000000 |

**TABLE 3.** probability for a specific unit to accomplish 30 CS start-ups in accurately 30 tried start-ups, 50 or fewer tried start-ups, 75 or fewer tried start-ups, 105 or fewer tried start-ups and 140 or more tried start-ups —— Supposing probability of singular successful start-up for p = 0.90, 0.95 and 0.99

| c = 30 | p = 0.90 | p = 0.95 | p = 0.99 |
|---|---|---|---|
| Accurately 30 Tried Start-Ups [P(30)] | 0.042391 | 0.214639 | 0.739700 |
| 50 or Fewer Tried Start-Ups [P(50)] | 0.127173 | 0.429278 | 0.887640 |
| 75 or Fewer Tried Start-Ups [P(75)] | 0.228569 | 0.650930 | 0.984747 |
| 105 or Fewer Tried Start-Ups [P(105)] | 0.334563 | 0.805012 | 0.998520 |
| 140 or More Tried Start-Ups [1-P(140)] | 0.439953 | 0.901064 | 0.999901 |

**TABLE 4.** probability for a specific unit to accomplish 50 CS start-ups in accurately 50 tried start-ups, 75 or fewer tried start-ups, 105 or fewer tried start-ups, 140 or fewer tried start-ups and 180 or more tried start-ups —— Supposing probability of singular successful start-up for p = 0.90, 0.95 and 0.99

| c = 50 | p = 0.90 | p = 0.95 | p = 0.99 |
|---|---|---|---|
| Accurately 50 Tried Start-Ups [P(50)] | 0.005154 | 0.076945 | 0.605006 |
| 75 or Fewer Tried Start-Ups [P(75)] | 0.018038 | 0.173126 | 0.756258 |
| 105 or Fewer Tried Start-Ups [P(105)] | 0.033484 | 0.286916 | 0.919092 |
| 140 or Fewer Tried Start-Ups [P(140)] | 0.051224 | 0.399811 | 0.974548 |
| 180 or More Tried Start-Ups [1-P(180)] | 0.069133 | 0.507360 | 0.993554 |

## 5. Conclusions

A special case happens when the demonstration test is to be truncated; in other words, the fire alarm systems are permitted just a specified entire number of tried start-ups to accomplish $c$ consecutive successes in advance. If $c$ consecutive successes do not happen to the trials of a stated entire number of tried start-ups in advance, the fire alarm system is rejected. The probability of acceptance is then only the probability of needing a pre-specified entire number or fewer tried start-ups to accomplish c consecutive successes.

Although we have the same results as the technique by Hahn and Gage [1], we used finite Markov chain imbedding method by Fu and Koutras [5] because of these several advantages.

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